THE CLIFF HANGER PROBLEM

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From where he stands, one step toward the cliff would send the drunken man over the bridge. He takes random steps, either toward or away from the cliff. At any step his probability of taking a step away is $\frac{2}{3}$, of a step toward the cliff $\frac{1}{3}$. What is his chance of escaping the cliff?

Suppose the coordinate for the cliff is at the origin and the drunken man is located at $x = 1$. Let the probability of stepping forward (toward the cliff) be p and step backward with probability $(1-p)$ accordingly. Let X_n be the probability that the drunken man will escape the cliff starting at $x = n$. It is obvious that $X_0 = 0$ and X_1 is what we want.

Brute-forth Approach: Based on the Bayes formula, we can get the difference equation

$$X_n = pX_{n-1} + (1-p)\,X_{n+1}, n \in \{1, 2, \cdots\}$$

Therefore we can derive the following

$$\begin{aligned}\sum_{i=1}^{n} X_i &= p\sum_{i=0}^{n-1} X_i + (1-p)\sum_{i=2}^{n+1} X_i \\ &= p\,(X_0 + X_1) + (p + 1 - p)\sum_{i=2}^{n-1} X_i + (1-p)\,(X_n + X_{n+1})\end{aligned}$$

Simplify the above, we get

$$(1-p)\,X_1 + pX_n = (1-p)\,X_{n+1}$$

therefore

$$X_{n+1} = \frac{p}{1-p}X_n + X_1$$

We already know that $\lim_{x\to\infty} X_{n+1} = \lim_{x\to\infty} X_n = 1$, hence we get

$$X_1 = 1 - \frac{p}{1-p} = \frac{1-2p}{1-p} = \frac{1}{2}$$

Approach Two: An elegant method

We consider the complement event, what is the chance of falling off the cliff for the drunken man? Similarly, let Y_n be the probability of falling off the cliff starting at $x = n$. It is straightforward $Y_0 = 1$. Again,

$$Y_1 = pY_0 + (1-p)\,Y_2$$

notice that Y_2 is the probability of falling the cliff, then the drunken man must ever go via the intermediate node $x = 1$ and then move toward the cliff and fall off. This implies that

$$Y_2 = Y_1^2$$

hence

$$Y_1 = p + (1-p)\,Y_1^2$$

Solve for X_1 we can get

$$Y_1 = 1, \frac{p}{1-p}$$

hence the valid solution is $Y_1 = \frac{p}{1-p}$ thus $X_1 = 1 - Y_1 = \frac{1-2p}{1-p}$.

Supplemental Calculation: I would like to get the general formula for X_n based on the above results. We know that

$$X_{n+1} = aX_n + b$$

where

$$a = \frac{p}{1-p}, b = X_1 = \frac{1-2p}{1-p}$$

Let $Z_{n+1} = \frac{X_{n+1}}{a^{n+1}}$ we simply get

$$\begin{aligned}
Z_{n+1} &= Z_n + \frac{b}{a^{n+1}} \\
&\vdots \\
Z_2 &= Z_1 + \frac{b}{a^2}
\end{aligned}$$

hence

$$\begin{aligned}
Z_n &= b\left(\sum_{i=1}^{n} \frac{1}{a^i}\right) \\
&= b\frac{\frac{1}{a^n}\left(1-a^n\right)}{1-a}
\end{aligned}$$

thus

$$X_n = a^n Z_n = b\frac{1-a^n}{1-a}$$

Therefore

$$X_n = \frac{1-2p}{1-p}\frac{1-\left(\frac{p}{1-p}\right)^n}{1-\frac{p}{1-p}} = 1 - \left(\frac{p}{1-p}\right)^n$$

which is consistent with the results for Y_n , i.e.,

$$Y_n = Y_1^n = \left(\frac{p}{1-p}\right)^n$$

which does make sense.

Approach 3: (Martingale Approach) The key idea is to find a martingale process. For the non-symmetric random walk, one can construct a process as such

$$M_n = \left(\frac{p}{q}\right)^{X_n}$$

where X_n is the coordinate at the n-th step, p is the probability of stepping toward the cliff ($X = 0$) and q is the probability of staying away from the cliff. Notice that

$$\begin{aligned}
\mathbb{E}\left[M_{n+1}\left|F_n\right.\right] &= \mathbb{E}\left[\left(\frac{p}{q}\right)^{X_{n+1}}\left|F_n\right.\right] \\
&= \mathbb{E}\left[\left(\frac{p}{q}\right)^{X_n+X}\left|F_n\right.\right] \\
&= \left(\frac{p}{q}\right)^{X_n}\mathbb{E}\left[\left(\frac{p}{q}\right)^{X}\right] \\
&= M_n\left[\left(\frac{p}{q}\right)^{-1}p + \left(\frac{p}{q}\right)^{1}q\right] \\
&= M_n
\end{aligned}$$

where F_n is the information available at n-th step and X is the random walk which can only take value ± 1. Therefore

$$\mathbb{E}\left[M_n\right] = \mathbb{E}\left[M_0\right] = \mathbb{E}\left[\left(\frac{p}{q}\right)^{X_0}\right] = \frac{p}{q}$$

Let $\tau = \inf\left\{n, X_n = 0\, or\, \infty\right\}$, a stopped martingale process that is stopped at stopping time is still a martingale process, therefore $\mathbb{E}\left[M_{n\wedge\tau}\right]$ is a martingale too. ($n \wedge \tau = \min\left\{n, \tau\right\}$by convention)

There are two absorbing states when $n \to \infty$, $X_{n\wedge\tau} = X_\tau = 0$ or $X_{n\wedge\tau} = X_\tau = \infty$. Based on the properties of a martingale process, we can get

$$\mathbb{E}\left[M_\tau\right] = \left(\frac{p}{q}\right)^0\mathbb{P}\left(X_\tau = 0\right) + \left(\frac{p}{q}\right)^\infty\mathbb{P}\left(X_\tau = \infty\right) = M_0 = \frac{p}{q}$$

In our case, $\frac{p}{q} < 1$ thus $\mathbb{P}\left(X_\tau = 0\right) = \frac{p}{q}$. In another words, the probability of escaping the cliff is $1 - \mathbb{P}\left(X_\tau = 0\right) = 1 - \frac{p}{1-p} = \frac{1-2p}{1-p}$

Keep in mind that $\frac{p}{q} < 1$, otherwise it is for sure (from a probability perspective) the drunken man will fall off the cliff.

A similar question based on difference equation is:

Urn R contains n red balls and urn B contains n blue balls. At each stage, a ball is selected at random from each urn, and they are swapped. Show that the mean number of red balls in urn R after stage k is $\frac{1}{2}n\left(1 + \left(1 - \frac{2}{n}\right)^k\right)$. This diffusion model was described by Daniel Bernoulli in 1769.

Let X_k be the number of red balls in urn R after stage k. Then the difference relation can be established as follows:

$$X_{k+1} = X_k + Y_k$$

where

$$Y_k = \begin{cases} 1 & \mathbb{P}\left(blue \longleftrightarrow red\right) = \frac{n-X_k}{n} \cdot \frac{n-X_k}{n} \\ 0 & \mathbb{P}\left(blue \longleftrightarrow blue\, or\, red \longleftrightarrow red\right) = \frac{n-X_k}{n} \cdot \frac{X_k}{n} + \frac{X_k}{n} \cdot \frac{n-X_k}{n} \\ -1 & \mathbb{P}\left(red \longleftrightarrow blue\right) = \frac{X_k}{n} \cdot \frac{X_k}{n} \end{cases}$$

Take expectation on both sides, we can get

$$\mathbb{E}\left[X_{k+1}\right] = \left(1 - \frac{2}{n}\right) \mathbb{E}\left[X_k\right] + 1$$

Solve for $\mathbb{E}\left[X_k\right]$ we get

$$\mathbb{E}\left[X_k\right] = \frac{1}{2} n \left(1 + \left(1 - \frac{2}{n}\right)^k\right)$$